The second IVP now is

$$s' = 3 - e^{-t}, \quad s(0) = 1,$$

whose solution, obtained in a similar manner, is

$$s(t) = 3t + e^{-t}. \ ■$$

1.21 Example. A stone is thrown upward from the ground with an initial speed of 39.2. To describe its motion when the air resistance is negligible, we need to establish a formula that gives the position $h(t)$ at time $t > 0$ of a heavy object moving vertically under the influence of the force of gravity alone. If $g = 9.8$ is the acceleration of gravity and the object starts moving from a height h_0 with initial velocity v_0, and if we assume that the vertical axis points upward, then Newton's second law yields the IVP

$$h'' = -g, \quad h(0) = h_0, \quad h'(0) = v_0.$$

Integrating the DE twice and using the ICs, we easily find that

$$h(t) = -\tfrac{1}{2}gt^2 + v_0t + h_0.$$

In our specific case, we have $g = 9.8$, $h_0 = 0$, and $v_0 = 39.2$, so

$$h(t) = -4.9t^2 + 39.2t.$$

If we now want, for example, to find the maximum height that the stone reaches above the ground, then we need to compute h at the moment when the stone's velocity is zero. Since

$$v(t) = h'(t) = -9.8t + 39.2,$$

we immediately see that v vanishes at $t = 4$, so

$$h_{\max} = h(4) = 78.4.$$

If, on the other hand, we want to know when the falling stone will hit the ground, then we need to determine the nonzero root t of the equation $h(t) = 0$, which, as can easily be seen, is $t = 8$. ■

Exercises

In **1–4**, solve the given IVP.

1 $y'' = -2(6t + 1); \ y(0) = 2, \ y'(0) = 0.$

2 $y'' = -12e^{2t}; \ y(0) = -3, \ y'(0) = -6.$

3 $y'' = -2\sin t - t\cos t; \ y(0) = 0, \ y'(0) = 1.$

4 $y'' = 2t^{-3}; \ y(1) = -1, \ y'(1) = 1.$

In **5–8**, solve the given BVP.

5 $y'' = 2, \ 0 < x < 1; \ y(0) = 2, \ y(1) = 0.$

6 $y'' = 4\cos(2x), \ 0 < x < \pi; \ y(0) = -3, \ y(\pi) = \pi - 3.$

7 $y'' = -x^{-1} - x^{-2}, \ 1 < x < e; \ y'(1) = 0, \ y(e) = 1 - e.$

8 $y'' = (2x - 3)e^{-x}, \ 0 < x < 1; \ y(0) = 1, \ y'(1) = -e^{-1}.$

In **9–12**, find the velocity $v(t)$ and position $s(t)$ at time $t > 0$ of a material particle that moves without friction along a straight line, when its acceleration, initial position, and initial velocity are as specified.

9 $a(t) = 2, \ s(0) = -4, \ v(0) = 0.$

10 $a(t) = -12\sin(2t), \ s(0) = 0, \ v(0) = 6.$

11 $a(t) = 3(t + 4)^{-1/2}, \ s(0) = 1, \ v(0) = -1.$

12 $a(t) = (t + 3)e^t, \ s(0) = 1, \ v(0) = 2.$

In **13** and **14**, solve the given problem.

13 A ball, thrown downward from the top of a building with an initial speed of 3.4, hits the ground with a speed of 23. How tall is the building if the acceleration of gravity is 9.8 and the air resistance is negligible?

14 A stone is thrown upward from the top of a tower, with an initial velocity of 98. Assuming that the height of the tower is 215.6, the acceleration of gravity is 9.8, and the air resistance is negligible, find the maximum height reached by the stone, the time when it passes the top of the tower on its way down, and the total time it has been traveling from launch until it hits the ground.

1.4 Classification of Differential Equations

We recall that, in calculus, a function is a correspondence between one set of numbers (called *domain*) and another set of numbers (called *range*), which has the property that it associates each number in the domain with exactly one number in the range. If the domain and range consist not of numbers but of functions, then this type of correspondence is called an *operator*. The image of a number t under a function f is denoted, as we already said, by $f(t)$; the image of a function y under an operator L is usually denoted by Ly. In special circumstances, we may also write $L(y)$.

1.22 Example. We can define an operator D that associates each differentiable function with its derivative; that is,

$$Dy = y'.$$

Naturally, D is referred to as the operator of differentiation. For twice differentiable functions, we can iterate this operator and write

$$D(Dy) = D^2y = y''.$$

This may be extended in the obvious way to higher-order derivatives. ■

1.23 Remark. Taking the above comments into account, we can write a differential equation in the generic form

$$Ly = f, \tag{1.1}$$

where L is defined by the sequence of operations performed on the unknown function y on the left-hand side, and f is a given function. We will use form (1.1) only in non-specific situations; in particular cases, this form is rather cumbersome and will be avoided. ■

1.24 Example. The DE in the population growth model (see Example 1.4) can be written as

$$P' - (\beta - \delta)P = 0.$$

In the operator notation (1.1), this is

$$LP = DP - (\beta - \delta)P = [D - (\beta - \delta)]P = 0;$$

in other words, $L = D - (\beta - \delta)$ and $f = 0$. ■

1.25 Example. It is not difficult to see that form (1.1) for the DE

$$t^2 y'' - 2y' = e^{-t}$$

is

$$Ly = (t^2 D^2)y - (2D)y = (t^2 D^2 - 2D)y = e^{-t},$$

so $L = t^2 D^2 - 2D$ and $f(t) = e^{-t}$. ■

1.26 Remarks. (i) The notation in the preceding two examples is not entirely rigorous. Thus, in the expression $D - (\beta - \delta)$ in Example 1.24, the first term is an operator and the second one is a function. However, we adopted this form because it is intuitively helpful.

(ii) A similar comment can be made about the term $t^2 D^2$ in Example 1.25, where t^2 is a function and D^2 is an operator. In this context, it must be noted that $t^2 D^2$ and $D^2 t^2$ are not the same operator. When applied to a function y, the former yields

$$(t^2 D^2)y = t^2(D^2 y) = t^2 y'',$$

whereas the latter generates the image

$$(D^2 t^2)y = D^2(t^2 y) = (t^2 y)'' = 2y + 4ty' + t^2 y''.$$

(iii) The rigorous definition of a mathematical operator is more general, abstract, and precise than the one given above, but it is beyond the scope of this book. ■

1.27 Definition. An operator L is called *linear* if for any two functions y_1 and y_2 in its domain and any two numbers c_1 and c_2 we have

$$L(c_1 y_1 + c_2 y_2) = c_1 L y_1 + c_2 L y_2. \qquad (1.2)$$

Otherwise, L is called *nonlinear*. ■

1.28 Example. The differentiation operator D is linear because for any differentiable functions y_1 and y_2 and any constants c_1 and c_2,

$$D(c_1 y_1 + c_2 y_2) = (c_1 y_1 + c_2 y_2)' = c_1 y_1' + c_2 y_2' = c_1 D y_1 + c_2 D y_2. \ ■$$

1.29 Example. The operator $L = tD^2 - 3$ is also linear, since

$$\begin{aligned} L(c_1 y_1 + c_2 y_2) &= (tD^2)(c_1 y_1 + c_2 y_2) - 3(c_1 y_1 + c_2 y_2) \\ &= t(c_1 y_1 + c_2 y_2)'' - 3(c_1 y_1 + c_2 y_2) \\ &= t(c_1 y_1'' + c_2 y_2'') - 3(c_1 y_1 + c_2 y_2) \\ &= c_1(t y_1'' - 3y_1) + c_2(t y_2'' - 3y_2) = c_1 L y_1 + c_2 L y_2. \ ■ \end{aligned}$$

1.30 Remark. By direct verification of property (1.2), it can be shown that, more generally,

(i) the operator D^n of differentiation of any order n is linear;

(ii) the operator of multiplication by a fixed function (in particular, a constant) is linear;

(iii) the sum of finitely many linear operators is a linear operator. ■

1.31 Example. According to the above remark, the operators written formally as

$$L = a(t)D + b(t), \quad L = D^2 + p(t)D + q(t)$$

with given functions a, b, p, and q, are linear. ■

1.32 Example. Let L be the operator defined by $Ly = yy'$. Then, taking, say, $y_1(t) = t$, $y_2(t) = t^2$, and $c_1 = c_2 = 1$, we have

$$\begin{aligned} L(c_1 y_1 + c_2 y_2) &= L(t + t^2) = (t + t^2)(t + t^2)' = t + 3t^2 + 2t^3, \\ c_1 L y_1 + c_2 L y_2 &= (t)(t)' + (t^2)(t^2)' = t + 2t^3, \end{aligned}$$

which shows that (1.2) does not hold for this particular choice of functions and numbers. Consequently, L is a nonlinear operator. ■

DEs can be placed into different categories according to certain relevant criteria. Here we list the most important ones, making reference to the generic form (1.1).

Number of independent variables. If the unknown is a function of a single independent variable, the DE is called an *ordinary differential equation.* If several independent variables are involved, then the DE is called a *partial differential equation.*

1.33 Example. The DE

$$ty'' - (t^2 - 1)y' + 2y = t \sin t$$

is an ordinary differential equation for the unknown function $y = y(t)$.

The DE

$$u_t - 3(x + t)u_{xx} = e^{2x - t}$$

is a partial differential equation for the unknown function $u = u(x, t)$. ■

Order. The order of a DE is the order of the highest derivative occurring in the expression Ly in (1.1).

1.34 Example. The equation

$$t^2 y'' - 2y' + (t - \cos t)y = 3$$

is a second-order DE. ■

Linearity. If the differential operator L in (1.1) is linear (see Definition 1.27), then the DE is a *linear equation*; otherwise it is a *nonlinear equation.*

1.35 Example. The equation

$$ty'' - 3y = t^2 - 1$$

is linear since the operator $L = tD^2 - 3$ defined by its left-hand side was shown in Example 1.29 to be linear. On the other hand, the equation

$$y'' + yy' - 2ty = 0$$

is nonlinear: as seen in Example 1.32, the term yy' defines a nonlinear operator. ■

Nature of coefficients. If the coefficients of y and its derivatives in every term in Ly are constant, the DE is said to be an *equation with constant coefficients*. If at least one of these coefficients is a prescribed non-constant function, we have an *equation with variable coefficients*.

1.36 Example. The DE

$$3y'' - 2y' + 4y = 0$$

is an equation with constant coefficients, whereas the DE

$$y' - (2t + 1)y = 1 - e^t$$

is an equation with variable coefficients. ■

Homogeneity. If $f = 0$ in (1.1), the DE is called *homogeneous*; otherwise it is called *nonhomogeneous*.

1.37 Example. The DE

$$(t - 2)y' - ty^2 = 0$$

is homogenous; the DE

$$y''' - e^{-t}y' + \sin y = 4t - 3$$

is nonhomogeneous. ■

Of course, any equation can be classified by means of all these criteria at the same time.

1.38 Example. The DE

$$y'' - y' + 2y = 0$$

is a linear, homogeneous, second-order ordinary differential equation with constant coefficients. The linearity of the operator $D^2 - D + 2$ defined by the left-hand side is easily verified. ■

1.39 Example. The DE

$$y''' - y'y'' + 4ty = 3$$

is a nonlinear, nonhomogeneous, third-order ordinary differential equation with variable coefficients. The nonlinearity is caused by the second term on the left-hand side. ■

1.40 Example. The DE

$$u_t + (x^2 - t)u_x - xtu = e^x \sin t$$

is a linear, nonhomogeneous, first-order partial differential equation with variable coefficients. ■

1.41 Definition. An ordinary differential equation of order n is said to be in *normal form* if it is written as

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)}), \qquad (1.3)$$

where F is some function of $n + 1$ variables. ■

1.42 Example. The equation

$$(t + 1)y'' - 2ty' + 4y = t + 3$$

is not in normal form. To write it in normal form, we solve for y'':

$$y'' = \frac{1}{t+1}(2ty' - 4y + t + 3) = F(t, y, y'). \;■$$

1.43 Definition. A function y defined on an open interval J (of the real line) is said to be a *solution* of the DE (1.3) on J if the derivatives $y', y'', \dots, y^{(n)}$ exist on J and (1.3) is satisfied at every point of J. ■

1.44 Remark. Sometimes a model is described by a *system* of DEs, which consists of several DEs for several unknown functions. ■

1.45 Example. The pair of equations

$$\begin{aligned} x_1' &= 3x_1 - 2x_2 + t, \\ x_2' &= -2x_1 + x_2 - e^t \end{aligned}$$

form a linear, nonhomogeneous, first-order system of ordinary DEs with constant coefficients for the unknown functions x_1 and x_2. ■

Exercises

Classify the given DE in terms of all the criteria listed in this section.

1 $y^{(4)} - ty'' + y^2 = 0$. **2** $y'' - 2y = \sin t$. **3** $y' - 2\sin y = t$.
4 $u_t - 2u_{xx} + (2xt + 1)u = 0$. **5** $2y''' + ye^y = 0$. **6** $uu_x - 2u_{xx} + 3u_{xxyy} = 1$.
7 $tu_t - 4u_x - u = x$. **8** $y'y''' - t^2u = \cos(2t)$.

Chapter 2
First-Order Equations

Certain types of first-order equations can be solved by relatively simple methods. Since, as seen in Sect. 1.2, many mathematical models are constructed with such equations, it is important to get familiarized with their solution procedures.

2.1 Separable Equations

These are equations of the form

$$\frac{dy}{dx} = f(x)g(y), \tag{2.1}$$

where f and g are given functions.

We notice that if there is any value y_0 such that $g(y_0) = 0$, then $y = y_0$ is a solution of (2.1). Since this is a constant function (that is, independent of x), we call it an *equilibrium solution*.

To find all the other (non-constant) solutions of the equation, we now assume that $g(y) \neq 0$. Applying the definition of the differential of y and using (2.1), we have

$$dy = y'(x)\,dx = \frac{dy}{dx}\,dx = f(x)g(y)\,dx,$$

which, after division by $g(y)$, becomes

$$\frac{1}{g(y)}\,dy = f(x)\,dx.$$

Next, we integrate each side with respect to its variable and arrive at the equality

$$G(y) = F(x) + C, \tag{2.2}$$

where F and G are any antiderivatives of f and $1/g$, respectively, and C is an arbitrary constant. For each value of C, (2.2) provides a connection between y and x, which defines a function $y = y(x)$ implicitly.

We have shown that every solution of (2.1) also satisfies (2.2). To confirm that these two equations are fully equivalent, we must also verify that, conversely, any function $y = y(x)$ satisfying (2.2) also satisfies (2.1). This is easily done by differentiating both sides of (2.2) with respect to x. The derivative of the right-hand side is $f(x)$; on the left-hand side, by the chain rule and bearing in mind that $G(y) = G(y(x))$, we have

C. Constanda, *Differential Equations*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-3-319-50224-3_2

$$\frac{d}{dx}\,G(y(x)) = \frac{d}{dy}\,G(y)\frac{dy}{dx} = \frac{1}{g(y)}\,\frac{dy}{dx},$$

which, when equated to $f(x)$, yields equation (2.1).

In some cases, the solution $y = y(x)$ can be determined explicitly from (2.2).

2.1 Remark. The above handling suggests that dy/dx could be treated formally as a ratio, but this would not be technically correct. ■

2.2 Example. Bringing the DE

$$y' + 8xy = 0$$

to the form

$$\frac{dy}{dx} = -8xy,$$

we see that it has the equilibrium solution $y = 0$. Then for $y \neq 0$,

$$\int \frac{dy}{y} = \int -8x\,dx,$$

from which

$$\ln|y| = -4x^2 + C,$$

where C is the amalgamation of the arbitrary constants of integration from both sides. Exponentiating, we get

$$|y| = e^{-4x^2+C} = e^C\,e^{-4x^2},$$

so

$$y(x) = \pm e^C e^{-4x^2} = C_1 e^{-4x^2}.$$

Here, as expected, C_1 is an arbitrary nonzero constant (it replaces $\pm e^C \neq 0$), which generates all the nonzero solutions y. However, if we allow C_1 to take the value 0 as well, then the above formula also captures the equilibrium solution $y = 0$ and, thus, becomes the GS of the given equation.

VERIFICATION WITH MATHEMATICA®. The input

```
y=C1*E^(-4*x^2);
D[y,x]+8*x*y
```

evaluates the difference between the left-hand and right-hand sides of our DE for the function y computed above. This procedure will be followed in all similar situations. As expected, the output here is 0, which confirms that this function is indeed the GS of the given equation.

The alternative coding

```
y[x_]=C1*E^(-4*x^2);
y'[x]+8*x*y[x]==0
```

gives the output `True`. Choosing one type of coding over the other is a matter of personal preference. Throughout the rest of the book we will use the former. ■

2.3 Example. In view of the properties of the exponential function, the DE in the IVP

$$y' + 4xe^{y-2x} = 0, \quad y(0) = 0$$

can be rewritten as

$$\frac{dy}{dx} = -4xe^{-2x}e^y,$$

and we see that, since $e^y \neq 0$ for any real value of y, the equation has no equilibrium solutions. After separating the variables, we arrive at

$$\int e^{-y}\,dy = -\int 4xe^{-2x}\,dx,$$

from which, using integration by parts (see Sect. B.2 in Appendix B) on the right-hand side, we find that

$$-e^{-y} = 2xe^{-2x} - \int 2e^{-2x}\,dx = (2x+1)e^{-2x} + C, \quad C = \text{const.}$$

We now change the signs of both sides, take logarithms, and produce the GS

$$y(x) = -\ln[-(2x+1)e^{-2x} - C].$$

The constant C is more easily computed if we apply the IC not to this explicit expression of y but to the equality immediately above it. The value is $C = -2$, so the solution of the IVP is

$$y(x) = -\ln[2 - (2x+1)e^{-2x}].$$

VERIFICATION WITH MATHEMATICA®. The input

```
y=-Log[2 - (2*x + 1)*E^(-2*x);
{D[y,x] + 4*x*E^(y - 2*x),y/.x→ 0}//Simplify
```

evaluates both the difference between the left-hand and right-hand sides (as in the preceding example) and the value of the computed function y at $x = 0$. Again, this type of verification will be performed for all IVPs and BVPs in the rest of the book with no further comment. Here, the output is, of course, $\{0, 0\}$. ■

2.4 Example. Form (2.1) for the DE of the IVP

$$xy' = y + 2, \quad y(1) = -1$$

is

$$\frac{dy}{dx} = \frac{y+2}{x}.$$

Clearly, $y = -2$ is an equilibrium solution. For $y \neq -2$ and $x \neq 0$, we separate the variables and arrive at

$$\int \frac{dy}{y+2} = \int \frac{dx}{x};$$

hence,

$$\ln|y+2| = \ln|x| + C, \quad C = \text{const},$$

from which, by exponentiation,

$$|y+2| = e^{\ln|x|+C} = e^C e^{\ln|x|} = e^C|x|.$$

This means that

$$y + 2 = \pm e^C x = C_1 x, \quad C_1 = \text{const} \neq 0,$$

so

$$y(x) = C_1 x - 2.$$

To make this the GS, we need to allow C_1 to be zero as well, which includes the equilibrium solution $y = -2$ in the above equality. Applying the IC, we now find that $C_1 = 1$; therefore, the solution of the IVP is

$$y(x) = x - 2.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y=x - 2;
{x*D[y,x] - y - 2, y/.x→ 1}//Simplify
```

generates the output $\{0, -1\}$. ■

2.5 Example. We easily see that the DE in the IVP

$$2(x+1)yy' - y^2 = 2, \quad y(5) = 2$$

has no equilibrium solutions; hence, for $x \neq -1$, we have

$$\int \frac{2y\,dy}{y^2+2} = \int \frac{dx}{x+1},$$

so

$$\ln(y^2+2) = \ln|x+1| + C, \quad C = \text{const},$$

which, after simple algebraic manipulation, leads to

$$y^2 = C_1(x+1) - 2, \quad C_1 = \text{const} \neq 0.$$

Applying the IC, we obtain $y^2 = x - 1$, or $y = \pm(x-1)^{1/2}$. However, the function with the '$-$' sign must be rejected because it does not satisfy the IC. In conclusion, the solution to our IVP is

$$y(x) = (x-1)^{1/2}.$$

If the IC were $y(5) = -2$, then the solution would be

$$y(x) = -(x-1)^{1/2}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y= (x - 1)^(1/2);
{2*(x + 1)*y*D[y,x] - y^2 - 2,y/.x→ 5}//Simplify
```

generates the output $\{0, 2\}$. ■

2.6 Example. Treating the DE in the IVP

$$(5y^4 + 3y^2 + e^y)y' = \cos x, \quad y(0) = 0$$

in the same way, we arrive at

$$\int (5y^4 + 3y^2 + e^y)\,dy = \int \cos x\,dx;$$

consequently,

$$y^5 + y^3 + e^y = \sin x + C, \quad C = \text{const}.$$

This equality describes the family of all the solution curves for the DE, representing its GS in implicit form. It cannot be solved explicitly for y.

The IC now yields $C = 1$, so the solution curve passing through the point $(0, 0)$ has equation

$$y^5 + y^3 + e^y = \sin x + 1.$$

Figure 2.1 shows the solution curves for $C = -2, -1, 0, 1, 2$. The heavier line (for $C = 1$) represents the solution of our IVP.

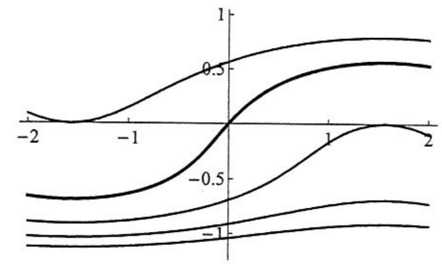

Fig. 2.1

VERIFICATION WITH MATHEMATICA®. The input

```
u=y[x]^5 + y[x]^3 + E^y[x] - Sin[x] - 1;
{(5*y[x]^4 + 3*y[x]^2 + E^y[x])*(Solve[D[u,x]==0, y'[x]])
      [[1,1,2]] - Cos[x], u/.{x→ 0, y→ 0}}//Simplify
```

generates the output $\{0, 0\}$, which shows that the function y defined implicitly above satisfies the DE and IC. ■

Exercises

Solve the given IVP.

1 $y' = -4xy^2$, $y(0) = 1$. **2** $y' = 8x^3/y$, $y(0) = -1$.

3 $y' = -6ye^{3x}$, $y(0) = e^{-2}$. **4** $y' = y\sin(2x)$, $y(\pi/4) = 1$.

5 $y' = (3-2x)y$, $y(2) = e^6$. **6** $y' = y^2e^{1-x}$, $y(1) = 1/3$.

7 $y' = (1+e^{-x})/(2y+2)$, $y(0) = -1$. **8** $y' = 2ye^{2x+1}$, $y(-1/2) = e^2$.

9 $y' = 2x\sec y$, $y(0) = \pi/6$. **10** $y' = 2x\sqrt{y}$, $y(1) = 0$.

11 $(1+2x)y' = 3+y$, $y(0) = -2$. **12** $3(x^2+2)y^2y' = 4x$, $y(1) = (\ln 9)^{1/3}$.

13 $y' = (6x^2+2x)/(2y+4)$, $y(1) = \sqrt{6} - 2$. **14** $(4-x^2)y' = 4y$, $y(0) = 1$.

15 $y' = (x-3)(y^2+1)$, $y(0) = 1$. **16** $2y(x^2+2x+6)^{1/2}y' = x+1$, $y(1) = -2$.

17 $y' = 2\sin(2x)/(4y^3+3y^2)$, $y(\pi/4) = 1$. **18** $y' = (2x+1)/(2y+\sin y)$, $y(0) = 0$.

19 $y' = (x^2+1)/(e^{-2y}+4y)$, $y(0) = 0$. **20** $y' = xe^{2x}/(y^4+2y)$, $y(0) = -1$.