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Endofinite modules over hereditary artinian PI-rings

Markus Schmidmeier

ABSTRACT. Let R be a hereditary artinian PI-ring. We describe the shape of the (connected) components of the AR-quiver Γ_R, which has as set of points a transversal of the indecomposable endofinite finite length R-modules. In particular we adapt the Theorem of Ringel and Auslander, Bautista, Platzeck, Reiten and Smalø to this more general situation.

"The representation theory of hereditary artin algebras is one of the most extensively studied and best understood theories developed to date." [**5**, from the introduction to chapter VIII]. What can be expected when extending the class of rings under consideration to hereditary artinian polynomial identity rings? From [**22**, Theorem 1] we know that the endofinite modules will play an important rôle in this theory: For a finitely generated indecomposable module M over an artinian PI-ring R there exists a source map $M \to D$ (respectively a sink map $B \to M$) in the category $\operatorname{Mod} R$ with D (respectively B) finitely generated if and only if M is endofinite. As a consequence, the structure of an Auslander-Reiten quiver Γ_R is defined on a transversal of the indecomposable endofinite finite length R-modules.

In this article we aim at a description of the shape of the (connected) components of the AR-quiver Γ_R for a hereditary artinian PI-ring R. In particular we will adapt the Theorem of Ringel [**16**] and Auslander, Bautista, Platzeck, Reiten and Smalø [**2**] to our situation.

THEOREM 1. *Let R be a hereditary artinian PI-ring and $\mathcal{C} \subset \Gamma_R$ a component which does not contain a projective or an injective module. Then $\mathcal{C}$ is quasi-serial, i. e. as a valued translation quiver, $\mathcal{C}$ is isomorphic to $\mathbb{Z}A_\infty$ or to a tube.*

Recall that the hereditary artinian rings of finite representation type and also their modules are completely classified, see [**10**]. For artinian tensor rings R with a duality condition whose underlying diagram is a Euclidean diagram, a description of the finite length R-modules is given by Dlab and Ringel [**7**] and Ringel [**15**]. The components of the AR-quiver Γ_R which contain a projective or an injective module are constructed for a large class of tensor rings R by Dowbor and Simson [**12**]. However, the example of Dlab and Ringel [**8**] shows that not every hereditary artinian PI-ring is a tensor ring.

The article is organized as follows. The examples in §1 show that the class of hereditary artinian PI-rings extends the class of hereditary artin algebras also in terms of a combinatorial datum; indeed, any valued quiver without oriented cycles can be realized by a hereditary artinian PI-ring. In the example in §2 we use the

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endolength to distinguish indecomposable modules of the same dimension vector. We introduce the endo-dimension vector and recall from [**21**] and [**22**] how the duality and the transpose manipulate the length and the endolength of a module. In §3 we show that a Morita duality preserves AR-sequences (in the category of all modules) which consist of finite length modules. The aim in §4 is to prove that the AR-quiver, which is defined on a transversal of the indecomposable endofinite finite length modules, has a homogeneous valuation. In §5 we will adapt the methods of [**5**] to recover the shape of those components of the AR-quiver of a hereditary artinian PI-ring which contain a projective module. Finally in §6 we prove Theorem 1 in such a way that we obtain as a byproduct a description of the shape of the regular components of the AR-quiver in the category of finite length modules in Zimmermann's example [**25**].

We use the following notation for a ring R. The category of all right R-modules is denoted by $\operatorname{Mod} R$ and we write $\operatorname{mod} R$ for the full subcategory of $\operatorname{Mod} R$ consisting of all *finite length* modules. We fix a transversal $\operatorname{ind} R$ of the indecomposable modules in $\operatorname{mod} R$ and denote by $\operatorname{mod}_{\text{ef}} R$ and $\operatorname{ind}_{\text{ef}} R$ the subclasses of endofinite modules in $\operatorname{mod} R$ and $\operatorname{ind} R$, respectively. Hom-sets are denoted as in $f \in (M_R, M'_R)$ or $g \in ({}_SN, {}_SN')$, morphisms are written on the opposite side of the scalars, as e. g. $f(m)$ or $(n)g$, and the composition is defined correspondingly. For further notation and basic results on module theory and representation theory we refer the reader to [**1**] and [**5**]. Besides the "usual" AR-sequences (in the category $\operatorname{Mod} R$), also *AR-sequences in the category* $\operatorname{mod} R$ will occur. These are nonsplit short exact sequences consisting of finite length modules with the factorization property of an AR-sequence restricted to test modules from $\operatorname{mod} R$.

§1. Hereditary artinian PI-rings

In this section we recall a characterization of artinian PI-rings and present a class of examples.

PROPOSITION 2 (A characterization of artinian PI-rings). *A semiprimary ring R, e. g. a onesided artinian ring, satisfies a polynomial identity if and only if the factor $\overline{R} = R/\operatorname{Rad} R$ modulo the (Jacobson-) radical is an artin algebra.* □

This result is well-known and can be shown easily using the theorems of Kaplansky and of Procesi and Small [**13**, 13.6.14 and 13.4.9]. In particular any artin algebra is an artinian PI-ring.

Note that the class of hereditary artinian PI-rings extends the class of hereditary artin algebras in terms of the following combinatorial datum.

Definition. A *(valued) quiver* $Q = (Q_0, Q_1, \mathrm{v}, \mathrm{v}')$ consists of a set Q_0 of points, a subset $Q_1 \subset Q_0 \times Q_0$ of arrows and two maps $\mathrm{v}, \mathrm{v}' : Q_1 \to \mathbb{N}$, the valuations. The *quiver* $Q(R) = (Q_0, Q_1, \mathrm{v}, \mathrm{v}')$ *of an artinian ring* R with radical $J = \operatorname{Rad} R$ is defined as follows: If $\{e_1, \ldots, e_n\}$ is a full set of pairwise nonisomorphic primitive idempotents of R, the set of points is $Q_0 = \{1, \ldots, n\}$. If $K(i) = e_iRe_i/e_iJe_i$ for $i \in Q_0$ and $B(\alpha) = e_{s(\alpha)}Je_{t(\alpha)}/e_{s(\alpha)}J^2e_{t(\alpha)}$ for $\alpha = (s(\alpha), t(\alpha)) \in Q_0 \times Q_0$, then the set of arrows is $Q_1 = \{\alpha \in Q_0 \times Q_0 : B(\alpha) \neq 0\}$, and the valuations v, $\mathrm{v}' : Q_1 \to \mathbb{N}$ are given by $\mathrm{v}(\alpha) = \dim_{K(s(\alpha))} B(\alpha)$ and $\mathrm{v}'(\alpha) = \dim B(\alpha)_{K(t(\alpha))}$.

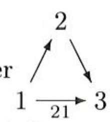
This notation deviates from e. g. [**7**] since we do not assume that there exist natural numbers f_i for $i \in Q_0$ such that $f_{s(\alpha)}\mathrm{v}(\alpha) = \mathrm{v}'(\alpha)f_{t(\alpha)}$ holds for all $\alpha \in Q_1$. Indeed, the existence of these numbers characterizes those finite quivers without oriented cycles which can be realized as quivers of a hereditary artin algebra [**5**, Prop. 6.4 and 6.7]. However, any finite quiver without oriented cycles can be realized as quiver of a hereditary artinian PI-ring. For this let k be a field and Q a finite quiver. If $K = k(X)$ is the field of rational functions with coefficients in k and n a natural number, the endomorphism $\phi^n : K \to K$ given by $\phi^n(X) = X^n$ and $\mathrm{Fix}\phi^n = k$ satisfies $\dim K_{\mathrm{Im}\phi^n} = n$, see e. g. [**23**, §63]. Define $T_k(Q)$ to be the tensor algebra $T(R_0, R_1)$, where R_0 is the ring $\prod_{i\in Q_0} K$ and R_1 the $R_0 - R_0$-bimodule $\prod_{\alpha\in Q_1} {}_{\phi^{\mathrm{v}(\alpha)}}K_{\phi^{\mathrm{v}'(\alpha)}}$ with the multiplication given by Q.

LEMMA 3. *Let Q be a finite quiver without oriented cycles and k a field. The tensor algebra $T_k(Q)$ is a hereditary artinian PI-ring with quiver Q.*

Proof. Since the K-tensor product of bimodules of type ${}_{\phi^n}K_{\phi^m}$ has finite K-dimension on either side, and since Q has no oriented cycles, the tensor ring $R = T_k(Q)$ is an artinian ring containing k in its centre. The ring R_0 is commutative, so R satisfies the identity $(YZ - ZY)^{|Q_0|} = 0$. Moreover we have for $i \in Q_0$

$$\mathrm{Rad}\, e_i R = \bigoplus_{\alpha\in Q_0,\ s(\alpha)=i} K_{\phi^{\mathrm{v}'(\alpha)}} \otimes_{R_0} R \cong \bigoplus_{\alpha\in Q_0,\ s(\alpha)=i} (e_{t(\alpha)}R)^{\mathrm{v}'(\alpha)},$$

where e_i is the primitive idempotent element corresponding to $i \in Q_0$. It follows that R is hereditary and has quiver Q. □

Example. Let Q be the quiver 

, which can not be realized by an artin algebra. The tensor ring $R = T_k(Q)$ is a hereditary artinian PI-ring. No nonzero R-module has finite length as a module over the centre k of R. In particular, the functor $\mathsf{D} = (-, {}_k k) : \mathrm{Mod}\, R \to R\,\mathrm{Mod}$ has the property that ${}_R\mathsf{D}M$ is not finitely generated for each nonzero module $M \in \mathrm{mod}\, R$. Nevertheless, the modules in $\mathrm{ind}_{\mathrm{ef}} R$ are accessible to methods of representation theory as we will see in the following sections.

§2. Endofinite modules

For the description of a finite length R-module M with endomorphism ring S the following numerical data are available.

$\ell(M_R)$	$\in$	$\mathbb{N}_0 \cup \{\infty\}$	the length of M_R
$[M_R]$	$\in$	$K_0(\mathrm{mod}\, R)$	the dimension vector if $\ell(M_R) < \infty$
$\varepsilon(M_R) = \ell({}_S M)$	$\in$	$\mathbb{N}_0 \cup \{\infty\}$	the endolength
$[{}_S M]$	$\in$	$K_0(S\,\mathrm{mod})$	the endo-dimension vector if $\varepsilon(M_R) < \infty$

In this section we give an example in which the endolength is used to distinguish indecomposable modules of the same dimension vector. We state a formula for the endo-dimension vector and recall how the duality manipulates the dimension vector and the endo-dimension vector.

Example. Let R be the ring in the Example in §1 and consider the decomposition $R_R = e_1R \oplus e_2R \oplus e_3R$ given by the quiver. Indecomposable right R-modules M of dimension vector $(1,1,1)$ can be obtained as cokernels M_x of maps $f_x : e_3R \to e_1R$, where $0 \neq x \in K \otimes K \oplus {}_{\phi^2}K$ denotes the image of e_3 in $\operatorname{Soc} e_1R$. To get a full list of the indecomposable modules of this dimension vector, up to isomorphism, we also include the cokernel $\tilde{M}$ of the map $\tilde{f} : e_3R \to e_2R \oplus e_1R/e_2R$, $e_3 \mapsto ((0,0,1), \overline{(0,0,(0,1))})$.

M	$\dim M$	$\operatorname{End} M$	$\varepsilon(M_R)$	layers in radical series of M
$M_{(0,1)}$	$(1,1,1)$	K	3	$1 \mid 2 \mid 3$
$M_{(1,0)}$	$(1,1,1)$	K	4	$1 \mid 23$
$\tilde{M}$	$(1,1,1)$	K	5	$12 \mid 3$
$M_{(a,1)},\ a \in K\backslash\{0\}$	$(1,1,1)$	k	∞	$1 \mid 2 \mid 3$

The endo-dimension vector of an endofinite module M can be considered as the tuple of the endolengths of the isoclasses of the indecomposable summands of M. More precisely, we have the following immediate consequence of [**6**, 4.5].

PROPOSITION 4 (A formula for the endo-dimension vector). *Let M_R be an endofinite module with endomorphism ring S and suppose that $M_R = M_1^{n_1} \oplus \cdots \oplus M_t^{n_t}$ is a decomposition such that the modules M_i are pairwise nonisomorphic and have local endomorphism ring. The simple $\operatorname{End} M_i^{n_i}$-modules X_i, $1 \leq i \leq t$, form a transversal of simple left S-modules. We have*

$$[{}_SM] = \sum_{i=1}^{t} \varepsilon(M_i)[{}_SX_i]$$

in $K_0(S\,\mathrm{mod})$. □

Now we recall some results about the operation of the duality and the transpose on finite length modules over an artinian ring R. Note first that due to the lack of a duality $\mathrm{mod}\,R \to R\,\mathrm{mod}$ given by the centre of the ring we will use the local duality L for the construction of AR-sequences. Indeed, by [**4**, I, Theorem 3.9] there exists for each nonprojective module $C \in \operatorname{ind} R$ an AR-sequence $0 \to \mathsf{L}\operatorname{Tr} C \to B \to C \to 0$ (in the category $\mathrm{Mod}\,R$), but the module $\mathsf{L}\operatorname{Tr} C$ may not have finite length.

Definition. For a module M_R with local endomorphism ring S the *local dual* is $\mathsf{L}M = {}_R({}_SM, {}_SI)$, where ${}_SI = \mathrm{E}({}_S\overline{S})$ is the injective envelope of the radical factor of S. For a finite sum $M = \coprod_i M_i$ of modules M_i with local endomorphism ring we put $\mathsf{L}M = \coprod_i \mathsf{L}M_i$. The local duality is not functorial in general.

To describe the local dual of a module we use the following two isomorphisms of Grothendieck groups. If R is a semiperfect ring, e. g. a semiprimary ring, we define

$$\lambda_R : K_0(\mathrm{mod}\,R) \to K_0(R\,\mathrm{mod}) \quad \text{by} \quad [\overline{eR}] \mapsto [\overline{Re}]$$

for primitive idempotents e. Suppose that S is a semiprimary ring and ${}_SI$ a finitely cogenerated injective cogenerator. Then also the endomorphism ring $T = \operatorname{End}{}_SI$ is a semiprimary ring and the functor $(-, {}_SI) : S\,\mathrm{mod} \to \mathrm{mod}\,T$ induces the isomorphism of Grothendieck groups

$$\mu_I : K_0(S\,\mathrm{mod}) \to K_0(\mathrm{mod}\,T), \quad [{}_SM] \mapsto [({}_SM, {}_SI)_T],$$

which coincides on the classes of the semisimple modules with the isomorphism of Grothendieck groups given by the Morita duality $(-, {}_S\mathrm{Soc}_S I) : \overline{S}\,\mathrm{mod} \to \mathrm{mod}\,\overline{T}$.

Definition. Using the notation of the previous definition, we say that M is L-*reflexive* if $(-, {}_S I)$ is a Morita duality with respect to which M is reflexive.

For a finite length module M over an artinian PI-ring R the following assertions are equivalent: (1) M is endofinite; (2) $\mathsf{L}M$ has finite length; (3) M is L-reflexive. Thus the endofiniteness determines the behavior of the local duality. We will also need the following quantitative result [**21**, Theorem 11].

THEOREM 5 (Dualizing modules over semiprimary PI-rings). *Let R and S be semiprimary PI-rings, ${}_S M_R$ a bimodule and ${}_S I$ a finitely cogenerated injective cogenerator with endomorphism ring $T = \mathrm{End}\,{}_S I$.*

1. *The ring T is a semiprimary PI-ring. The module ${}_S M$ has finite length if and only if the dual module $({}_S M, {}_S I)_T$ has finite length. In this case, we have $[({}_S M, {}_S I)_T] = \mu_I[{}_S M]$ in $K_0(\mathrm{mod}\,T)$.*
2. *If two of the modules ${}_S M$, M_R, ${}_R({}_S M, {}_S I)$ have finite length, all three have finite length. If this is the case and if ${}_S I = \mathrm{E}({}_S\overline{S})$, we have $[{}_R({}_S M, {}_S I)] = \lambda_R[M_R]$ in $K_0(R\,\mathrm{mod})$.*
3. *Suppose ${}_S I = \mathrm{E}({}_S\overline{S})$ and $J_T = \mathrm{E}(\overline{T}_T)$. Both ${}_S M$ and M_R have finite length if and only if the bidual module $(({}_S M, {}_S I)_T, J_T)_R$ has finite length.* □

Also recall that the transpose preserves and reflects endofiniteness for finite length modules over artinian PI-rings [**22**].

§3. Morita duality

In this section we show that a Morita duality preserves endofiniteness for finite length modules over an artinian PI-ring R. As a consequence we obtain that the Morita dual of an AR-sequence (in $R\,\mathrm{Mod}$) consisting of finite length modules is again an AR-sequence (in the category of all modules). In preparation of the next sections we also study the commutativity of LM and $\mathsf{M}\,\mathrm{Tr}$.

PROPOSITION 6 (On the Morita duality). *Let R be an artinian PI-ring and ${}_R Q$ a finitely cogenerated injective cogenerator with endomorphism ring R'.*

1. *The functor $\mathsf{M} = (-, {}_R Q) : R\,\mathrm{mod} \to \mathrm{mod}\,R'$ is a Morita duality. For $M \in R\,\mathrm{mod}$ we have $[(\mathsf{M}M)_{R'}] = \mu_Q[{}_R M]$.*
2. *If $M \in R\,\mathrm{mod}$ has endomorphism ring S, then $\mathsf{M}M$ becomes a left S-module using the canonical isomorphism $S \cong \mathrm{End}(\mathsf{M}M)_{R'}$. The module ${}_R M$ is endofinite if and only if the Morita dual module $(\mathsf{M}M)_{R'}$ is endofinite. If this is the case and if ${}_R Q = \mathrm{E}({}_R\overline{R})$, then $[{}_S\mathsf{M}M] = \lambda_S[M_S]$ in $K_0(S\,\mathrm{mod})$.*
3. *The ring R' is an artinian PI-ring, moreover, R' is hereditary if and only if R is hereditary.*
4. *If $\mathcal{E}$ is an AR-sequence in $R\,\mathrm{Mod}$ consisting of finite length modules, then $\mathsf{M}\mathcal{E}$ is an AR-sequence in $\mathrm{Mod}\,R'$ consisting of finite length modules.*

Proof. 1. According to Rosenberg and Zelinsky's theorem [**19**, Theorem 3], ${}_R Q$ is finitely generated, hence induces a Morita duality [**1**, Theorem 30.4].

2. This is a consequence of 2. in Theorem 5 (with the rôle of R and S exchanged).

3. As Morita dual of a ring, R' is right artinian. As endomorphism ring of a finitely generated module over a PI-ring, R' itself is a PI-ring. We get from [**27**, Prop. 3] or from Theorem 5 (put $M = R$ and $I = Q$ in 1.) that ${}_RQ$ is endofinite. The second assertion of this proposition implies that $R' = \mathsf{M}Q$ is left artinian.

4. If $\mathcal{E}$ is an AR-sequence in $R\,\mathrm{Mod}$ consisting of finite length modules, the dual sequence $\mathsf{M}\mathcal{E}$ is an AR-sequence in $\mathrm{mod}\,R'$. By [**22**, Theorem 1] (quoted in the introduction) the last term of $\mathcal{E}$ is endofinite. We get from the second assertion that the first term of $\mathsf{M}\mathcal{E}$ is endofinite, in particular this module is pure injective, so $\mathsf{M}\mathcal{E}$ is an AR-sequence in $\mathrm{Mod}\,R'$ by [**24**, Prop. 3]. □

Next we will show that LM commutes with $\mathsf{M}\,\mathrm{Tr}$.

PROPOSITION 7. *Let R be an artinian PI-ring with Morita dualities $\mathsf{M} = (-, {}_RQ) : R\,\mathrm{mod} \to \mathrm{mod}\,R'$ and $\mathsf{M} : \mathrm{mod}\,R \to R''\,\mathrm{mod}$. We also denote the inverses of these Morita dualities by M.*

1. *The functor $\mathsf{M}\,\mathrm{Tr} : \mathrm{mod}\,R/\mathcal{P}(\mathrm{mod}\,R) \to \mathrm{mod}\,R'/\mathcal{I}(\mathrm{mod}\,R')$ is an equivalence of categories, where $\mathcal{P}(\mathrm{mod}\,R)$ (respectively $\mathcal{I}(\mathrm{mod}\,R)$) denotes the categorial ideal consisting of those homomorphisms which factor through a projective (respectively an injective) module. If R is also hereditary, $\mathsf{M}\,\mathrm{Tr}$ is a functor $\mathrm{mod}\,R \to \mathrm{mod}\,R'$ which preserves monomorphisms and those epimorphisms whose kernel does not have a projective summand.*
2. *The map $\mathsf{LM} : \mathrm{mod}_{\mathrm{ef}}R' \to \mathrm{mod}_{\mathrm{ef}}R$ preserves and reflects simple, injective and projective modules. For $M \in \mathrm{mod}_{\mathrm{ef}}R'$ we have $[(\mathsf{LM}M)_R] = \lambda_R\mu_Q[M_{R'}]$.*
3. *The following diagram commutes pointwise up to isomorphism.*

$$\begin{array}{ccc} \mathrm{mod}_{\mathrm{ef}}R & \xrightarrow{\mathsf{M}\,\mathrm{Tr}} & \mathrm{mod}_{\mathrm{ef}}R' \\ {\scriptstyle \mathsf{LM}}\downarrow & & \downarrow{\scriptstyle \mathsf{LM}} \\ \mathrm{mod}_{\mathrm{ef}}R'' & \xrightarrow[\mathsf{M}\,\mathrm{Tr}]{} & \mathrm{mod}_{\mathrm{ef}}R \end{array}$$

4. *For $n \in \mathbb{N}$ there are artinian PI-rings $R_0 = R, R_1 = R', \ldots, R_n$ such that there exist Morita dualities $\mathsf{M} : R_i\,\mathrm{mod} \to \mathrm{mod}\,R_{i+1}$, $0 \le i < n$. For $M \in \mathrm{mod}_{\mathrm{ef}}R$ we have $(\mathsf{L}\,\mathrm{Tr})^n M \cong (\mathsf{LM})^n(\mathsf{M}\,\mathrm{Tr})^n M$.*

Proof. 1. This assertion is shown as for modules over artin algebras.

2. It is well-known that the local duality maps simple modules to simple modules, indecomposable projective modules to indecomposable injective modules and – in this case – indecomposable injective modules to indecomposable projective modules (see e. g. [**21**, Theorem 2]).

3. It suffices to check the commutativity of the diagram for nonprojective modules $C \in \mathrm{ind}_{\mathrm{ef}}R$. Therefore we take an AR-sequence $0 \to A \to B \to C \to 0$ in $\mathrm{Mod}\,R$ and show that $A = \mathsf{L}\,\mathrm{Tr}\,C \cong (\mathsf{M}\,\mathrm{Tr})(\mathsf{LM})C$. By Proposition 6, the sequence $0 \to \mathsf{M}C \to \mathsf{M}B \to \mathsf{M}A \to 0$ is an AR-sequence in $\mathrm{Mod}\,R$, hence $\mathsf{M}C \cong \mathsf{L}\,\mathrm{Tr}\,\mathsf{M}A$. The module $\mathrm{Tr}\,\mathsf{M}A$ is as an endofinite module L-reflexive (Proposition 6 and the notes after and before Theorem 5), so applications of L, Tr and M yield the isomorphism $(\mathsf{M}\,\mathrm{Tr})(\mathsf{LM})C \cong A$.

4. This assertion follows from 1. in Proposition 6 and 3. □

§4. Auslander-Reiten quivers

Let R be an artinian PI-ring and Γ_0 the set $\mathrm{ind}_{\mathrm{ef}} R$. Up to isomorphism, this set contains all simple modules, all indecomposable projective modules since R_R is endofinite and since summands of endofinite modules also are endofinite [**6**, Prop. 4.4], and all indecomposable injective modules, which are finitely generated by [**19**, Theorem 3] and endofinite as dual modules of projective modules [**27**, Prop. 3]. In this section the AR-quiver Γ_R (in the category $\mathrm{Mod}\, R$) is defined in the usual way as a translation quiver on the set of points Γ_0. We will show that this quiver has *homogeneous valuation*, i. e. the translation preserves the valuations of the arrows.

Define the set of arrows Γ_1 as the set of those pairs $\alpha = (s(\alpha), t(\alpha)) \in \Gamma_0 \times \Gamma_0$ for which there exists a morphism $s(\alpha) \to t(\alpha)$ which is irreducible in the category $\mathrm{Mod}\, R$. Then (Γ_0, Γ_1) is an oriented graph without loops. We know from [**22**, Theorems 2 and 1] that Γ_0 is closed under irreducible predecessors and successors and that for each $M \in \Gamma_0$ there exist a sink map $M \to D$ and a source map $B \to M$ in the category $\mathrm{Mod}\, R$ with B and D finite length modules.

Using the construction of irreducible maps from sink maps or source maps [**3**, Theorem 2.4] we obtain the following valuations on Γ_1.

$$\begin{aligned} d(\alpha) &= \max\{n \in \mathbb{N} \,:\, s(\alpha)^n \text{ is isomorphic to a summand of } B\}, \\ &\quad\ \text{where } B \to t(\alpha) \text{ is a sink map in the category } \mathrm{Mod}\, R \\ d'(\alpha) &= \max\{n \in \mathbb{N} \,:\, t(\alpha)^n \text{ is isomorphic to a summand of } D\}, \\ &\quad\ \text{where } s(\alpha) \to D \text{ is a source map in the category } \mathrm{Mod}\, R \end{aligned}$$

It follows from the characterization of AR-sequences by sink maps and source maps [**4**, II, Prop. 4.4] that there is a bijection $\tau : \Gamma_0 \backslash \mathcal{P} \to \Gamma_0 \backslash \mathcal{I}$ such that for a module $M \in \Gamma_0 \backslash \mathcal{P}$ the set M^- of Γ_1-predecessors of M coincides with the set $(\tau M)^+$ of Γ_1-successors of τM. Here $\mathcal{P}$ and $\mathcal{I}$ denote the set of all projective and all injective modules in Γ_0, respectively. The *AR-quiver (in the category* $\mathrm{Mod}\, R$*)* is the translation quiver

$$\Gamma_R = (\Gamma_0, \Gamma_1, d, d', \tau).$$

Remark. Besides the AR-quiver in the category $\mathrm{Mod}\, R$ we will also consider as Example 1 in §6 an AR-quiver in the category $\mathrm{mod}\, R$ for the ring R studied in [**25**]. There it is shown that for each nonprojective module C in $\mathrm{ind}\, R$ (respectively each noninjective module A in $\mathrm{ind}\, R$) there exists an AR-sequence $0 \to A \to B \to C \to 0$ in the category $\mathrm{mod}\, R$. Sink maps in $\mathrm{mod}\, R$, source maps in $\mathrm{mod}\, R$ and irreducible homomorphisms in $\mathrm{mod}\, R$ are defined in the canonical way and satisfy the relations mentioned above. Thus for this particular ring R, the *AR-quiver in the category* $\mathrm{mod}\, R$ can be constructed analogically on the set $\Gamma_0 = \mathrm{ind}\, R$.

In the remainder of this section we show that the AR-quiver (in $\mathrm{Mod}\, R$) has homogeneous valuation for an artinian PI-ring R.

Let $\mathcal{E} : 0 \to A \to \coprod_i B_i^{m_i} \to C \to 0$ be an AR-sequence such that the modules B_i are pairwise nonisomorphic and have local endomorphism ring. It follows from the axioms of a source map and from its uniqueness [**3**, Prop. 2.2] that the canonical map $\zeta : K_A \to K_C$ is an isomorphism of skew fields, where K_A denotes the radical factor of $\mathrm{End}\, A$. The following bilinear form is known from [**17**, (3.2) and (3.4)], cf. also [**9**].

Lemma 8. *There is a nondegenerate homomorphism of* $K_{B_i} - K_{B_i}$*-bimodules*

$$\phi_i : \operatorname{irr}(A, B_i) \otimes_{K_A} {}_\zeta\operatorname{irr}(B_i, C) \to K_{B_i},$$

where $\operatorname{irr}(A, B_i)$ *denotes the* $K_{B_i} - K_A$*-bimodule* $\operatorname{rad}(A_R, B_{iR}) / \operatorname{rad}^2(A_R, B_{iR})$. □

From the next lemma we obtain the following isomorphism of $K_{B_i} - K_A$-bimodules.

$$\operatorname{irr}(A, B_i) \;\cong\; (\operatorname{irr}(B_i, C)_{K_{B_i}}, K_{B_i})_\zeta$$

Lemma 9 (On the Hom-Tensor adjoint isomorphism). *Let* R, S *and* T *be rings and* ${}_RX_S$, ${}_SY_T$ *and* ${}_RZ_T$ *be bimodules.*

1. *A homomorphism* $\phi \in (X \otimes_S Y_T, Z_T)$ *is an* $R - T$*-bimodule homomorphism if and only if the adjoint homomorphism* $\psi : X_S \to (Y_T, Z_T)_S$ *given by* $\psi(x)(y) = \phi(x \otimes y)$ *is an* $R - S$*-bimodule homomorphism.*
2. *Let* ${}_RX$ *be a finitely presented module and* Z_T *an injective cogenerator which is left balanced, i. e. every endomorphism of* Z_T *is given by left multiplication. The map* ϕ *in the first assertion is a left and right nondegenerate bimodule homomorphism if and only if* ψ *is a bimodule isomorphism.*

Proof. The first assertion is verified easily. For the proof of the second let $\phi : X \otimes_S Y \to Z$ be an $R - T$-bimodule homomorphism. It is easy to see that ϕ is left nondegenerate if and only if ψ is a monomorphism. If ψ is surjective, then ϕ is right nondegenerate: For any $0 \neq y \in Y$ there is $f \in (Y_T, Z_T)$ such that $f(y) \neq 0$ since Z_T is a cogenerator. Now our assumption implies that there is an $x \in X$ such that $f = \phi(x \otimes -)$.

Conversely if ϕ is right nondegenerate, then the map $\chi : Y_T \to ({}_RX, {}_RZ)_T$ given by $\chi(y) = (x \mapsto \phi(x \otimes y))$ is a monomorphism. Hence for $f : Y_T \to Z_T$ there is $g : ({}_RX, {}_RZ)_T \to Z_T$ such that $f = g\chi$ since Z_T is injective. Since ${}_RX$ is finitely presented and Z_T is injective, the canonical map $(Z_T, Z_T) \otimes_R X \to (({}_RX, {}_RZ)_T, Z_T)$ is an isomorphism and we get $\zeta_i \in \operatorname{End} Z_T$ and $x_i \in X$, $i = 1, \ldots, n$, such that $g(u) = \sum_{i=1}^n \zeta_i((x_i)u)$ for $u \in ({}_RX, {}_RZ)$. Since Z is left balanced, g is the evaluation at some $x \in X$. Hence $f(y) = \phi(x \otimes y)$ for $y \in Y$. Thus, ψ is surjective. □

Proposition 10. *The AR-quiver of an artinian PI-ring has homogeneous valuation.*

Proof. Let R be an artinian PI-ring with AR-quiver $\Gamma_R = (\Gamma_0, \Gamma_1, d, d', \tau)$. Suppose $\mathcal{E}$ as above is an AR-sequence in $\operatorname{Mod} R$ consisting of finite length modules. Since the endomorphism ring of a finite length module is semiprimary [**1**, Cor. 29.3], and since the endomorphism ring of a finitely generated module over a PI-ring also is a PI-ring [**13**, 13.4.9], we obtain from Proposition 2 that the skew fields K_{B_i} and K_C are finite dimensional over their centres. With an application of the Lemma of Dowbor and Simson [**11**, Prop. 1.3] we can compute the right and left dimension of the dual module in the isomorphism of bimodules constructed with Lemma 9.

$$\dim \operatorname{irr}(A, B_i)_{K_A} = \dim {}_{K_C}\operatorname{irr}(B_i, C), \quad \dim {}_{K_{B_i}}\operatorname{irr}(A, B_i) = \dim \operatorname{irr}(B_i, C)_{K_{B_i}}$$

For $\alpha \in \Gamma_1$ it can be shown as in [**5**, VIII, Prop. 1.3] that $d(\alpha)$ and $d'(\alpha)$ coincide with the right and left dimension of the $K_{t(\alpha)} - K_{s(\alpha)}$-bimodule $\operatorname{irr}(s(\alpha), t(\alpha))$, respectively. If $s(\alpha)$ and $t(\alpha)$ are nonprojective modules we have just seen that these dimensions coincide with the left and right dimension of the $K_{s(\alpha)} - K_{\tau t(\alpha)}$-bimodule

$\operatorname{irr}(\tau t(\alpha), s(\alpha))$ and also with the right and left dimension of the $K_{\tau t(\alpha)} - K_{\tau s(\alpha)}$-bimodule $\operatorname{irr}(\tau s(\alpha), \tau t(\alpha))$, respectively. This completes the proof. □

§5. Preprojective components in the hereditary case

Throughout this section let R be a hereditary artinian PI-ring. The indecomposable projective R-modules are endofinite, so we can study their component(s) in the AR-quiver. This has been done for a large class of tensorrings in [**12**], but not every hereditary artinian PI-ring is a tensorring [**8**]. We will follow the approach in [**5**, VIII, §1-2] replacing whenever necessary the duality given by the centre of R by a Morita duality or by the local duality. The proofs of the following results are given in greater detail in [**20**]. Also the corresponding statements about components of the AR-quiver containing injective modules hold.

Definition. Let $E_1, \dots, E_n$ be simple right R-modules in a 1-1-correspondence to the points in the quiver of R. Let P_i be a projective hull and I_i an injective envelope of E_i for $i = 1, \dots, n$. We may choose the modules E_i, P_i and I_i as elements of $\operatorname{ind}_{\mathrm{ef}} R$. Since R has finite global dimension, the sets $\{[E_i] : i = 1, \dots, n\}$, $\{[P_i] : i = 1, \dots, n\}$ and $\{[I_i] : i = 1, \dots, n\}$ are bases of $K_0(\operatorname{mod} R)$. Hence the *coxeter transformation* $\mathrm{c} : K_0(\operatorname{mod} R) \to K_0(\operatorname{mod} R)$, $[P_i] \mapsto -[I_i] = -[\mathsf{L}P_i^*]$, is defined and an isomorphism of groups. An element $m \in K_0(\operatorname{mod} R)$ is called *positive* (respectively *negative*) if all coordinates of m with respect to the basis $\{[E_i] : i = 1, \dots, n\}$ are positive (respectively negative).

PROPOSITION 11 (On the coxeter transformation). *For $M \in \operatorname{mod} R$ we have*

$$\mathrm{c}[M] = \lambda_R[\operatorname{Tr} M] - \lambda_R[M^*].$$

In particular if M is indecomposable, then $\mathrm{c}[M]$ is either negative or positive; $\mathrm{c}[M]$ is negative if and only if M is projective. If M is endofinite, then $\mathrm{c}[M] = [\mathsf{L}\operatorname{Tr} M] - [\mathsf{L}M^*]$.

Proof. For the first assertion apply the functor $(-, R_R)$ to a minimal projective presentation of M_R. Now multiply the classes in $K_0(R\operatorname{mod})$ of the modules in the long exact sequence by λ_R and use 2. in Theorem 5 to compute $\mathrm{c}[M]$. The further statements are immediate consequences. □

Definition. A module $M \in \operatorname{ind} R$ is *preprojective* if there exists a $t \in \mathbb{N}_0$ such that $(\mathsf{L}\operatorname{Tr})^i M$ is a finite length module for $i = 1, \dots, t$ and $(\mathsf{L}\operatorname{Tr})^t M$ is an indecomposable projective module. In this case we put $\mathrm{p}(M) = t$ and, if $(\mathsf{L}\operatorname{Tr})^t M \cong P_j$, $\mathrm{P}(M) = j$. A component of Γ_R is *preprojective* if it consists of preprojective modules and does not contain oriented cycles.

PROPOSITION 12 (Preprojective modules and preprojective components).

1. *A module $M \in \operatorname{ind} R$ is preprojective if and only if $\mathrm{c}^n[M]$ is negative for some $n \in \mathbb{N}_0$.*
2. *Suppose that $M, M' \in \operatorname{ind} R$ satisfy $[M_R] = [M'_R]$. If M is preprojective, then $M_R \cong M'$.*
3. *Any component of Γ_R which contains a projective module is a preprojective component.*

Proof. Let $M \in \operatorname{ind} R$ be a nonprojective module. Note that in Proposition 11 we have constructed a module $M' \in \operatorname{ind} R$ with $[M'] = \mathrm{c}[M]$ only in the situation that M is an endofinite module. In order to study an iterated application $\mathrm{c}^n[M]$ of the coxeter transformation we construct Morita dualities $\mathsf{M} : R_i \operatorname{mod} \to \operatorname{mod} R_{i+1}$ for $i = 0, \dots, n-1$ with $R_0 = R$ and consider the modules $(\mathsf{M}\,\mathrm{Tr})^i M \in \operatorname{mod} R_i$ for $i = 0, \dots, n$. Using Proposition 7 and Proposition 11 one can controll the dimension vectors of these modules and detect projectivity. With this modification, the results in the proposition can be shown as the corresponding results in [**5**]. □

Now we can read the shape of the preprojective components in Γ_R from the composition structure of R.

PROPOSITION 13 (The shape of the preprojective components). *Let $\mathcal{C}$ be the union of all preprojective components of Γ_R. There is an injective morphism of valued translation quivers*

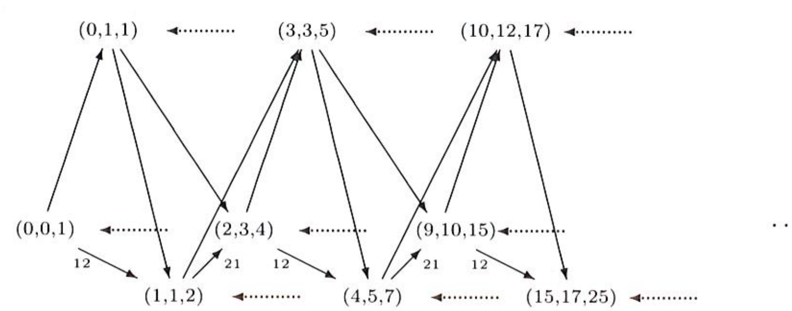
$$\beta : \mathcal{C} \to \mathbb{N}_0 Q(R)^{\mathrm{op}}, \quad M \mapsto (\mathrm{p}(M), \mathrm{P}(M)).$$

Proof. This proposition can be shown as the corresponding statement for artin algebras, [**5**, VIII, Prop. 1.15]. Only for the proof that β preserves both valuations of arrows between projective vertices one may want to use the following result. □

LEMMA 14. *A homomorphism $p : P \to Q$ between finitely generated projective R-modules is irreducible in the category* $\operatorname{Mod} R$ *if and only if $p^* : Q^* \to P^*$ is irreducible in the category* $R \operatorname{Mod}$.

Proof. Since $*$ induces a duality between the full subcategories of $\operatorname{mod} R$ and $R \operatorname{mod}$ consisting of projective modules, p is split if and only if p^* is split. We only show the direction "⇒". Assume that $p^* = gh$ for some $Y \in R \operatorname{Mod}$, $g : Q^* \to Y$ and $h : Y \to P^*$. Put $g' = gh : Q^* \to \operatorname{Im} h$ and $h' = \mathrm{incl} : \operatorname{Im} h \to P^*$. Since R is hereditary, $\operatorname{Im} h$ is a finitely generated projective module. Since p is irreducible, so is p^{**} and g'^* or h'^* splits. Hence h' is a split epimorphism or g' is a split monomorphism. This implies that h is a split epimorphism or g is a split monomorphism. □

We conclude this section by sketching the preprojective component of Γ_R where R is the ring from the example in §1.



§6. Regular components

A component $\mathcal{C}$ of an AR-quiver is called *regular* if there is no projective and no injective module in $\mathcal{C}$. In this section we describe the shape of the regular components of the AR-quiver of a hereditary artinian PI-ring (Theorem 1) and consider two examples.

The left exactness of the functor D Tr seems to be a key ingredient in the proofs of the corresponding result for hereditary artin algebras in [**16**], [**2**] and [**18**]. However, if the underlying ring is not an artin algebra, the functoriality of the local duality L, which replaces D, may not be available.

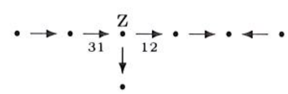
So we establish a criterion for a component $\mathcal{C}$ of an AR-quiver to be quasi-serial which uses only the following *weak left exactness condition* for the AR-translate τ.

$$(*)\quad \begin{cases} \text{For } n, m \in \mathbb{N} \text{ and every monomorphism } f : X \to Y^m \\ \text{with } X, Y \in \mathcal{C} \text{ we have } \ell(\tau^n X) \le m \cdot \ell(\tau^n Y) \end{cases}$$

We state our criterion in such a way that it can be applied also to study components of an AR-quiver in the category of finite length modules.

PROPOSITION 15 (A test for quasi-serial). *Let R be a right artinian ring and $\mathcal{M}_R$ either of the categories* Mod R *or* mod R. *Suppose that the AR-quiver Γ_R in the category $\mathcal{M}_R$ exists and that $\mathcal{C} \subset \Gamma_R$ is a regular component with homogeneous valuation. If condition $(*)$ is satisfied, then $\mathcal{C}$ is a quasi-serial component.*

Proof. This proof follows the steps in the proof of [**18**, Theorem]. Recall that a *star with centre z* is a finite quiver $Q = (Q_0, Q_1, \mathrm{v}, \mathrm{v}')$ such that the underlying nonoriented graph of Q is a star with centre z and such that each arrow $\alpha \in Q_1$ satisfies $\mathrm{v}(\alpha) = 1$ if $t(\alpha) \neq z$ and $\mathrm{v}'(\alpha) = 1$ if $s(\alpha) \neq z$. A star Q is said to have n *rays* if $n = \sum_{\{\alpha\in Q_1 : t(\alpha)=z\}} \mathrm{v}(\alpha) + \sum_{\{\alpha\in Q_1 : s(\alpha)=z\}} \mathrm{v}'(\alpha)$. For example, the following quiver is a star with 6 rays.



Suppose that $\mathcal{C}$ is a regular component of Γ_R with homogeneous valuation. From [**14**, Struktursatz] we obtain a tree Q, i. e. a connected oriented graph without unoriented cycles, and a covering $\pi : \mathbb{Z}Q \to \mathcal{C}$ of translation quivers without valuation. Take the unique valuation (d, d') on $\mathbb{Z}Q$ such that π becomes a morphism of (valued) translation quivers. Since the valuation on $\mathcal{C}$ is homogeneous, there exists a valuation on Q which induces this valuation on $\mathbb{Z}Q$.

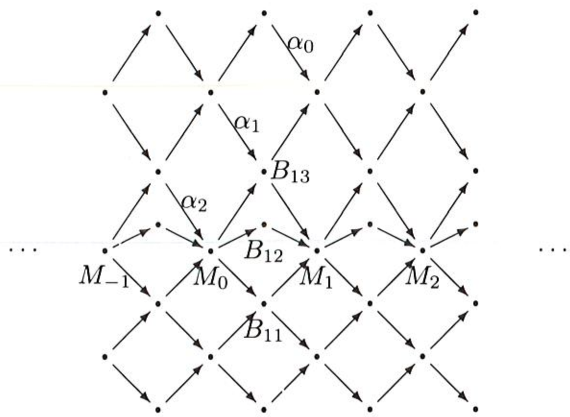
We can obtain detailed information about this quiver Q using the length function $\ell : (\mathbb{Z}Q)_0 \to \mathbb{N}, x \mapsto \ell(\pi x_R)$, which has the following properties. It is

- *additive*, i. e. $\ell(y) + \ell(\tau y) = \sum_{x\in y^-} d(x, y) \cdot \ell(x)$ for $y \in (\mathbb{Z}Q)_0$, since there is a short exact sequence $0 \to \pi\tau y \to \coprod_{x\in y^-} (\pi x)^{d(x,y)} \to \pi y \to 0$ in mod R,
- *strict*, i. e. $\ell(x) \neq \ell(y)$ for $x \to y \in (\mathbb{Z}Q)_1$, since there exists an irreducible morphism $\pi x \to \pi y$ in the category $\mathcal{M}_R$ and
- *monotonic*, i. e. $\ell(\tau x) \le \ell(\tau y)$ for all $x \to y \in (\mathbb{Z}Q)_1$ satisfying $\ell(x) \le \ell(y)$, since we have the weak left exactness condition $(*)$.

By [**18**, Prop.] the following assertions are equivalent for a connected quiver Q.

i) There exists an additive, strict and monotonic function $\ell : (\mathbb{Z}Q)_0 \to \mathbb{N}$.
ii) The quiver Q is A_∞ or a star, but not a Dynkin diagram.

It remains to exclude the case that Q is a star. Therefore we assume that Q is a star with centre z and n rays and construct a contradiction. To visualize the situation we draw a picture (with Q a quiver of type $\tilde{E}_7$).



The arrows $\alpha_i \in \mathbb{Z}Q_1$ pointing towards the centre correspond to irreducible monomorphisms in $\mathcal{M}_R$: This can be shown successively using a length argument and the following AR-sequences.

$$0 \longrightarrow \pi s(\alpha_{i+1}) \longrightarrow \pi t(\alpha_{i+1}) \oplus \pi s(\alpha_i) \longrightarrow \pi t(\alpha_i) \longrightarrow 0$$

Put $M_i = \pi(i, z)$ for $i \in \mathbb{Z}$. From the AR-sequences

$$0 \longrightarrow M_{i-1} \longrightarrow \coprod_{j=1}^{n} B_{ij} \longrightarrow M_i \longrightarrow 0,$$

in which the indecomposable summands B_{ij} of the middle term occur with a multiplicity given by the quiver, we hence obtain monomorphisms

$$f_i : M_{i-1} \longrightarrow M_i^n.$$

Thus we have for each $t \in \mathbb{N}$ a monomorphism $g_t = f_t^{(n^{t-1})} \circ \cdots \circ f_2^n \circ f_1 : M_0 \longrightarrow M_t^{(n^t)}$, and if $m = \ell(M_{0R})$, there is also a monomorphism $h_t : M_0 \longrightarrow M_t^m$.

Using the weak left exactness condition $(*)$ again, we obtain for each $t \in \mathbb{N}$ an inequality $\ell(M_{-t}) \leq m \cdot \ell(M_0) = m^2$. Hence we have that all $x \in \mathbb{Z}Q$ which lie on the left hand side of $(0, z)$ satisfy $\ell(x) \leq m^2$. There exists a nonzero homomorphism $P \longrightarrow M_0$ from a projective module P. This homomorphism can be factored through a sum of compositions of $2^{(m^2)} - 1$ irreducible maps between indecomposable modules of length at most m^2 — we obtain a contradiction from the Lemma of Harada and Sai [**5**, VI, Cor. 1.3] to our assumption that Q is a star. Thus the quiver Q is A_∞, and we have shown that $\mathcal{C}$ is a quasi-serial component. □

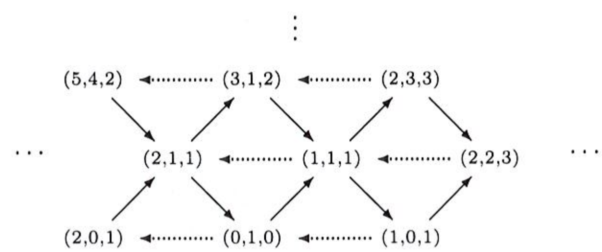
We are now ready to prove our Theorem on the shape of the regular components.

Proof of Theorem 1. Let R be a hereditary artinian PI-ring and $\mathcal{C}$ a regular component of the AR-quiver Γ_R, which has a homogeneous valuation according to Proposition 10. We show that the weak left exactness condition $(*)$ in Proposition 15 holds. Decompose the map $(\mathsf{L}\,\mathrm{Tr})^n : \mathrm{mod}_{\mathrm{ef}} R \longrightarrow \mathrm{mod}_{\mathrm{ef}} R$ into $(\mathsf{LM})^n \circ (\mathsf{M}\,\mathrm{Tr})^n$ as in Proposition 7. Since $(\mathsf{M}\,\mathrm{Tr})^n : \mathrm{mod}\, R \longrightarrow \mathrm{mod}\, R'$ is a left exact functor and since the isomorphism of Grothendieck groups $K_0(\mathrm{mod}\, R') \longrightarrow K_0(\mathrm{mod}\, R)$ induced by $(\mathsf{LM})^n$ preserves the classes of simple modules, we conclude that the AR-translate τ is weakly left exact. □

Example 1. In [**25**] and [**26**] Zimmermann has studied the existence of AR-sequences in the category mod R for the following hereditary artinian PI-ring R. Let K be the field of formal Laurant series in one variable over some field k of characteristic zero, $\delta : K \to K$ a certain derivative and ${}_K B = K \oplus K$ the $K-K$-bimodule with right multiplication $(b, b')c = (bc, b'c + b\,\delta(c))$. Put $R = \begin{pmatrix} K & B \\ 0 & K \end{pmatrix}$. It is shown that any module $M \in \operatorname{ind} R$ which is neither preprojective nor preinjective occurs in an AR-sequence in the category mod R of type $\mathcal{E} : 0 \to M \to B \to M \to 0$. Indeed, M is endofinite if and only if $\mathcal{E}$ is an AR-sequence in the category Mod R.

We obtain the following description of the AR-quiver Γ_R in the category mod R, as defined in the Remark in §4. Since AR-sequences in Mod R which consist of finite length modules are also AR-sequences in mod R, the results in §5 hold also for the preprojective component and the preinjective component of Γ_R. From Proposition 15 we obtain that all the regular components of Γ_R are quasi-serial; it follows from the particular form of the AR-sequences above that the regular components are more precisely tubes of diameter 1. In the family of these tubes there is one tube which consists of endofinite modules according to [**15**, Theorem 4]. The modules in the other tubes can be considered as torsion modules over the derivation polynomial ring $K[X;\delta]$, their components are parametrized by the equivalence classes of the irreducible polynomials in $K[X;\delta]$. None of these modules is endofinite; more precisely it is shown in [**25**, Proof of Theorem 13 and Theorem 15] that the endomorphism ring of such a module is a finite dimensional k-algebra.

Example 2. Let the ring R be as in the Example in §1. Since the characteristic polynomial of the coxeter matrix of R has three real roots different from 1, all regular components of Γ_R must be of type $\mathbb{Z}A_\infty$. According to §2 there are, up to isomorphism, three indecomposable endofinite modules of dimension vector $(1,1,1)$, which have endolength 3, 4 and 5. The modules of endolength 3 and 5 occur as quasi-simple modules in their components; the component containing the module of dimension vector $(1,1,1)$ and endolength 4 has the following shape.



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KATEDRA ALGEBRY, MFF - UK, SOKOLOVSKÁ 83, 18600 PRAHA 8, CZECH REPUBLIC
E-mail address: mschmidm@karlin.mff.cuni.cz